A Strange Induction

**Problem:** Prove that for any sequence \( \{a_m\} \) such that

\[
gcd(a_n + 2, a_n + 1) > a_n.
\]

the condition \( a_n \geq 2^n \) holds.

**Solution:** For the benefit of the reader I first give a few examples and the thought process that went into finding this solution. Also, for illustration of the problem, note that the sequence

1, 2, 4, 8, 16, . . .

does satisfy this recurrence and has the strict equality \( a_n = 2^n \).

We felt that the natural approach to a problem like this was an induction. This seems to be a good technique because the sequence is defined recursively and we are trying to prove something is true for all \( n \). Direct induction seems not to work very well however because these sequences can look very strange locally. Here is one example that deflates most inductions

1, 10, 11, 33, 66, . . .

As you can see the change on a local scale things don’t behave nicely because a large early jump can lead to strange behavior later. Seeing possible difficulties we chose rather to use an argument by contradiction. But we can make one important description

\( a_n \) is monotone strictly increasing

**Proof:** Suppose \( a_i \geq a_{i+1} \). But by the recursion we also have \( a_{i+1} \geq \gcd(a_{i+2}, a_{i+1}) > a_i \) so we have a contradiction. ■

Now suppose \( a_k \) is the smallest element in the sequence such that \( a_k < 2^k \). Our goal will be to show that this cannot be the case. We do this by showing that each element \( a_i \) must be within its own interval \( 2^i \leq a_i < 2^{i+1} \). By assumption we already have \( 2^i \leq a_i \) for all \( i < k \).

**First Induction** \( a_i < 2^{i+1} \).

We induce downwards from \( a_k \). Since our sequence is monotonically increasing we have \( a_{k-1} < a_k < 2^k \) so our base case is done. Now we assume \( a_{i+1} < 2^{i+2} \). Then we have

\[
2^{i+1} > \frac{a_{i+1}}{2} \geq \gcd(a_{i+2}, a_{i+1}) > a_i
\]

Now that we have put all of the \( a_i \) securely in its own interval we can show that if this is the case we must have \( a_i = 2^{i+1} \) and then to a contradiction because we know how that sequence must go.

**Second Induction** \( a_i = 2^{i+1} \)

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*Date: June 21, 2011.*