Holiday problems 2010-2011

Geometry

Hexagons

Divide a hexagon into equilateral triangles, as in the figure. Now fill the hexagon with the three kinds of oriented diamonds made from two triangles, also shown in the figure. Prove that the number of each kind of diamond is the same.

Shadows

A cylinder of length $h$ and radius $r$ is randomly situated above the $xy$ plane (so that all orientations are equally likely), on which it vertically casts a shadow. What is the expected area of that shadow? Generalize!

Squares

Prove that if you take a unit square and cut it into a finite number of smaller squares the side lengths of the smaller squares are all rational.

Limits

There is only one pair of positive integers $A$ and $B$ for which

$$\lim_{x \to 0} \frac{\sinh (\sin (x)) - \sin (\sinh (x))}{x^4} = \frac{1}{B}.$$  

What is this pair of integers?

Series

(1) Show that $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$ converges, but not absolutely.

(2) Can you find a continuous periodic function $f$ (of period $2\pi$), not identically zero, for which $\sum_{n=1}^{\infty} \frac{f(n)}{n}$ converges?
Numbers

Sum representations

Succinctly characterize (i.e., give a formula for) the whole numbers that cannot be written as the sum of two or more consecutive whole numbers.

For example, 3 = 1 + 2, 5 = 2 + 3, and 6 = 1 + 2 + 3 can be written as such sums whereas 1, 2, and 4 obviously cannot.

Sum-product partitions

For $n \geq 5$ it is possible to partition the numbers $\{1, 2, \ldots, n\}$ into two disjoint subsets such that the product of the elements in one set equals the sum of the elements in the other. One solution is to use $\left\{1, \frac{n-2}{2}, n\right\}$ and its complement when $n$ is even and otherwise use $\left\{1, \frac{n-1}{2}, n-1\right\}$ and its complement.

Is this solution unique for infinitely many $n$?

Boxes and Cakes

On the table before you is a cylindrical ice-cream cake with chocolate icing on top. From it you cut successive radial wedges of angle $x$, where $x$ is arbitrary. Each time a wedge is cut, it is turned upside-down and reinserted into the cake. Prove that after a finite number of such operations, all the icing is back on top of the cake (Winkler, p. 111).

NB: $x$ can be an irrational fraction of $2\pi$. The result is still true!

Strange Cubic

Let $\alpha = 1 + a_1x + a_2x^2 + \ldots$ be a formal power series with coefficients in the field of two elements. Let

$$a_n = 1 \quad \text{if every block of zeros in the binary expansion of } n \text{ has an even number of zeros in the block,}$$

$$a_n = 0 \quad \text{otherwise.}$$

(For example, $a_{36} = 1$ because $36 = 100100_2$ and $a_{20} = 0$ because $20 = 10100_2$.) Prove that $\alpha^3 + x\alpha + 1 = 0$. (Putnam 1989 A6.)