Solution Set 1

At last week's meeting we tackled the first two calculus problems (Problem Set 1). The first concerned a runner who traversed 6 miles in 30 minutes. We were to prove there was some five-minute period in which she ran exactly one mile.

The key was to write down the data in a helpful way. We decided to let 't' denote time elapsed since the start of the race, 0 <= t <= 30, and x(t) denote the position arrived at by time t. Thus x(30) - x(0) = 6 miles. We don't know anything else about this function x() except that it is continuous (no instantaneous teleportation is allowed in a race!). If we want to, we may assume it's differentiable (she has a well-defined speed at every time), but it turns out this was not needed.

With this notation, the distance traversed during the last five minutes is z(t) = x(t) - x(t-5), provided t >= 5, of course. We need to show there is some t for which z(t) = 1 exactly. Because z() is formed by a combination of constants, subtraction, and composition out of a continuous function (x, that is), we know that z() itself is continuous. Moreover, z(5) + z(10) + ... + z(30) = [x(5) - x(0)] + [x(10) - x(5)] + ... + [x(30) - x(25)] = x(30) - x(0) = 6. Therefore it is impossible for all six of the numbers z(5), ..., z(30) to be less than 1 (their sum would be less than 6) and it is equally impossible for all six to be greater than 1. Suppose, then, that t0 is a time at which z(t0) >= 1 and t1 is a time at which z(t1) <= 1. Since z is continuous, the Intermediate Value Theorem assures us there is some time t between t0 and t1 for which z(t) = 1. QED.

The second problem asked us to maximize the function 2^(-x) + 2^(-1/x), x > 0. We found it very helpful to use "logarithmic derivatives." This is the idea that the derivative of the logarithm of any positive function f(x) is f'(x)/f(x). Multiplying out by f(x) gives this extremely useful formula:

f'(x) = f(x) * d/dx ln(f(x)).

That looks nastier than it is. It is usually applied when the log of f() is much simpler than f() itself. For instance, to differentiate exp(-x^2), just differentiate its log

d/dx (-x^2) = -2x

and multiply by f:

d/dx exp(-x^2) = -2x * exp(-x^2).

That was trivial, but now try something like 2^(-1/x):

d/dx 2^(-1/x) = 2^(-1/x) * d/dx ln(2^(-1/x)).

But by the laws of logarithms, ln(2^(-1/x)) = -1/x * ln(2), so its derivative is easy to find: it's 1/x^2 * ln(2). Therefore, d/dx 2^(-1/x) = 2^(-1/x) / x^2 * ln(2).

This didn't really help us much in analyzing our function, though: the derivative is still too messy. The breakthrough came when we combined two observations:

(a) Converting x to 1/x does not change our function.
(b) Taking logarithms, converting x to 1/x changes ln(x) to -ln(x).

In other words, if we write ln(x) = y, we will have a nice symmetric function. This makes life easier. (Moral: look for and exploit symmetries in problem situations.)

Here we go: if ln(x) = y, y = exp(x). The function to work with is

2^(-x) + 2^(-1/x)
= 2^(-exp(y)) + 2^(-exp(-y)).

This is an even function of y: it has the same value at -y as it has at y. Thus, in order to find maxima, we can focus on y >= 0. It's easy to see that when y = 0, x = 1, so the function's value is 2^-1 + 2^-1/1 = 1/2 + 1/2 = 1. Almost as easy is that as y becomes large, x becomes large, 2^-x approaches zero, and 2^-1/x approaches 2^0 = 1. Thus the function approaches a value of 1 asymptotically as y increases.

To see how the function behaves between zero and infinity, we need to find the critical points. We do this by logarithmically differentiating the two terms separately:

d/dx 2^(-exp(y)) = 2^(-exp(y)) * -exp(y) * ln(2)
d/dx 2^(-exp(-y)) = 2^(-exp(-y)) * exp(-y) * ln(2)

(Be careful with those minus signs!) Evidently, then, by factoring out ln(2), critical points occur when

2^(-exp(y)) * exp(y) = 2^(-exp(-y)) * exp(-y).

Taking the terms involving 2^something to one side and all others to the other side gives

2^(exp(y) - exp(-y)) = exp(y + y) = exp(2y).

This will be true if and only if it's true for the logarithms:

[exp(y) - exp(-y)] * ln(2) = 2y;
[exp(y) - exp(-y)]/2 = y/ln(2);
sinh(y) = y/ln(2).

There's no closed solution to this equation, but we can find all solutions graphically. We found exactly two solutions for y >= 0. One occurs at y = 0 and the other occurs at the unique intersection of the line f(y) = y/ln(2) with the curve f(y) = sinh(y). We know there's a unique intersection because the derivative of sinh is cosh, cosh(0) = 1, the derivative of cosh is sinh, and sinh(y) > 0 for all y > 0. These facts tell us that the line defined by y/ln(2) increases faster than sinh(y) at y = 0, because 1/ln(2) > 1, and they tell us that sinh() is convex for y >= 0. Therefore there can be at most one positive root of sinh(y) - y/ln(2) and--because sinh(y) grows like exp(y)/2--there has to be a root.

Conclusion: there is exactly one critical point of 2^(-x) + 2^(-1/x) for x > 1. It's either a local max or local min. We decided it's a local min because a value near this point, at x = 2, is 2^-2 + 2^(-1/2) = 1/4 + sqrt(1/2). We checked that this value is less than 1, since (1 - (1/4))^2 = 9/16 > 1/2 = sqrt(1/2)^2. Because the *unique* positive critical point is a local min, the asymptotic values (at x --> 0 and x --> infinity) are local suprema, equal to 1, and the remaining critical point at x = 1 is a local max, also equal to 1. That solves the problem.

The two features of this problem that made it more difficult than your run-of-the-mill calculus textbook problem were:

(1) Differentiating it required some skill and
(2) It was not possible explicitly to find the zeros of the derivative.

Nevertheless, mechanically this problem was routine. One key was to find and exploit a symmetry and a change of variable from x to y = ln(x) that reduced the computational work we did. Another was to recognize a (somewhat) familiar function, the hyperbolic sine, allowing us to exploit its nice properties. (We didn't really _need_ to know about sinh, but working directly with [exp(x) - exp(-x)]/2 would have taken more effort.) These provide yet more examples of the Principle of Mathematical Laziness ("do as little work as possible").

* * * * * * * * *

The other two calculus problems remain unsolved! If you think you have a solution, please share it with the rest of us by e-mail. They are both interesting problems with interesting and useful methods of solution, so they're worth some thinking time. Here are some hints:

#3 (Covering a graph with n rectangles): Use induction.

#4 (Limiting value of a recursive sequence): (a) [Clever method] Use the Principle of Mathematical Laziness to pick "nice" starting values X0, X1. (There are several ways to do this.) What limits result? Can you find a relationship between the general situation and these "nice" situations? (b) [Even cleverer method] Another approach is to find starting values X0, X1 for which the sequence takes on a particularly simple form (such as being constant, or alternating in value, or something like that whose limit is really simple to handle). Can you express the general limit in terms of these "simple" sequences? (c) [Generating function method] Form the power series having X(n)/n! as its coefficients. Write down the second-order differential equation satisfied by this function. Solve it. Compute the Taylor series for the solution you obtain. (d) [Linear algebra method] Write the recursion in the form (X(n), X(n+1)) = (X(n-1), X(n))*A for some 2 x 2 matrix A. Find a matrix B such that B^-1*A*B is diagonal. In either (c) or (d), use your results to find a general (and explicit) formula for X(n) and take the limit.