Problem Set 6: Inequalities

Inequalities abound in mathematics, but proving them can be notoriously difficult. Arthur Engel, in his book *Problem-Solving Strategies*, presents a startling number of simple but effective techniques. (He proves one particular inequality in eleven distinct ways!) What follows is a synopsis of part of his exposition with an example or two for you to work on to practice each technique. The inequalities in **bold** are worth knowing well and memorizing: they are useful in many different ways. Consider these to be warm-ups for more challenging problems to come!

1. Perhaps the most basic inequality is that $x^2 \geq 0$ with equality if and only if $x = 0$.
   1.a Prove that $(x^2 + 2)/\sqrt{(x^2 + 1)} \geq 2$.
   1.b For all real $a, b, c$, $a^2 + b^2 + c^2 \geq ab + bc + ca$.

2. A very effective one is the **Arithmetic-Geometric Mean (AM-GM) Inequality**: the arithmetic mean of $n$ positive real numbers is never less than their geometric mean. For two numbers $x$ and $y$ this says $\sqrt{xy} \leq (x + y)/2$.
   2.a Prove the AM-GM inequality.
   2.b For non-negative real numbers $a, b, c$, show that $(a + b)(b + c)(c + a) \geq 8abc$.
   2.c If the product of $n$ positive real numbers $a_1, \ldots, a_n$ is 1, then $(1+a_1)(1+a_2)\ldots(1+a_n) \geq 2^n$.
   2.d Use the AM-GM inequality to prove 1.b.

3. Prove the Arithmetic Mean-Harmonic Mean (AM-HM) inequality: the sum of $n$ positive real numbers, multiplied by the sum of their reciprocals, is never less than $n^2$.

4. Prove the **Cauchy-Schwartz Inequality**: the inner product of two real $n$-vectors is less than or equal to the product of the norms of those vectors, with equality if and only if the vectors are parallel. In symbols, let the vectors be $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. Then $a_1b_1 + \ldots + a_nb_n \leq \sqrt{(a_1^2 + \ldots + a_n^2)} \sqrt{(b_1^2 + \ldots + b_n^2)}$ with equality if and only if there exists a real number $c$ such that $b_1 = ca_1$, ..., and $b_n = ca_n$ (or vice versa). Hint: find the point on the line through $a$ in the direction $b$ which is closest to the origin.
   4.a Prove the Arithmetic Mean-Quadratic Mean inequality, $(a_1 + \ldots + a_n)/n \leq \sqrt[(n)]{(a_1^2 + \ldots + a_n^2)/n}$, by means of the Cauchy-Schwartz inequality.

5. Here is one that is perfectly obvious but can be used to prove a wide variety of complicated looking inequalities: the **Rearrangement Inequality**. Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be equal-length sequences of positive real numbers. Then the sum of products $a_1b_1 + \ldots + a_nb_n$ is maximized over all permutations of $b$ when the $a$’s and $b$’s
are in the same sorted order and it is minimized when the $a$'s and $b$'s have been sorted in opposite order.

5.a Prove this.

5.b $a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a$ for positive $a$, $b$, and $c$.

5.c Use the sequences $(a, b, c)$ and $(1/(b+c), 1/(c+a), 1/(a+b))$ to prove Nesbitt's Inequality $(a/(b+c) + b/(c+a) + c/(a+b)) \geq 3/2$. To accomplish this, you may assume $(a, b, c)$ is sorted, in which case $(1/(b+c), 1/(c+a), 1/(a+b))$ is also in the same order. Apply the Rearrangement Inequality to the two cyclic permutations of $(1/(b+c), 1/(c+a), 1/(a+b))$ and add the two results.

5.d Find the minimum of $\sin^3(x)/\cos(x) + \cos^3(x)/\sin(x)$ for $x$ strictly between 0 and $\pi/2$.

6. Convexity. Recall that a differentiable function is convex at a point when its second derivative is positive there. (There is a more general definition applying to non-differentiable functions.) **Jensen's Inequality** says that whenever $f$ is a convex function of a single real variable, then any weighted mean of $f(x_1), \ldots, f(x_n)$ is greater than or equal to $f$ applied to the weighted mean of its arguments. That is, if we take the weights $w_1, \ldots, w_n$ to be real numbers between 0 and 1 and summing to 1, then $w_1 f(x_1) + \ldots + w_n f(x_n) \geq f(w_1 x_1 + \ldots + w_n x_n)$. This is a very general and powerful tool. It follows that convex functions attain their maxima only at extremal points of their domains. (An extremal point is one that cannot be written as a nontrivial weighted average of two or more distinct points.)

6.a Prove the AM-HM inequality by applying Jensen’s Inequality to the function $f(x) = 1/x$.

6.b Prove the AM-GM inequality by applying Jensen’s Inequality to the function $f(x) = \ln(x)$.

6.c What inequalities result when you apply Jensen’s Inequality to the functions $f(x) = x^p$? (Which values of $p$ even give convex functions?)

6.d If $a$, $b$, and $c$ lie between 0 and 1, then $a/(b+c+1) + b/(c+a+1) + c/(a+b+1) + (1-a)(1-b)(1-c) \leq 1$. Hint: show, by taking second derivatives, that this function of $(a, b, c)$ is convex within its domain (a cube). Therefore, the inequality only has to be checked at the eight vertices $(0,0,0)$, $(1,0,0)$, $\ldots$, $(1,1,1)$ of the cube, where it is trivial.

6.e Prove Jensen’s Inequality.