Invariants and Symmetry

Terminology

An invariant is some property of an object that does not change even when the object is changed or its description is changed.

A semi-invariant is a property that changes in a predictable way as a function of the change.

A monovariant is an ordered property that changes in a predictable way (usually by increasing or decreasing) as a sequence of changes is wrought upon an object.

Principles

Problems always become simpler when you can exploit invariants or symmetries.

When a group acts on a set, the properties of an orbit are an invariant of the orbit’s members. Whence, invariants are often constructed by adding (or combining in some similar manner) an object and its images under all elements of a group. This principle shows there is a close connection between symmetries and invariants. In practice then, when symmetries are evident, use them to construct invariants; when sums of related things are in evidence, try to uncover the symmetries for which the sums are invariants.

Coefficients of a polynomial are invariant under permutations of its roots.

Examples

[1] (Eötvös Competition 1906/3; Zeitz 3.4.8) Let $a_1, a_2, \ldots, a_n$ represent an arbitrary arrangement of the numbers $1, 2, 3, \ldots, n$. Prove that, if $n$ is odd, the product

$$(a_1 - 1)(a_2 - 2)(a_3 - 3) \ldots (a_n - n)$$

is an even number.

Solution. What stays the same (is invariant) when the $a$’s are permuted? According to our principles above, an easy invariant is their sum. Thus, the sum of the factors in the product $(a_1 - 1) + (a_2 - 2) + (a_3 - 3) + \ldots + (a_n - n)$ also is invariant: indeed, it’s zero, because it simply adds the numbers 1 through $n$ in a permuted order (the $a$’s) and then subtracts them again. If all the factors were odd, though, their sum would have to be odd (because $n$ is odd). Therefore at least one factor is even, making the entire product even.

[2] Sum the geometric series $S = 1 + r + r^2 + \ldots + r^n + \ldots$ for $|r| < 1$.

Solution. This series looks like an invariant constructed from an infinite sequence of operations. Indeed, given any number $r$, the natural numbers act on the real numbers through multiplication by $r$. The orbit of 1 consists of $\{1, r, r^2, \ldots, r^n, \ldots\}$ and this sum (which converges only for $|r| < 1$) is almost an invariant of the orbit. (If the natural
numbers formed a group under addition, the sum would automatically be an invariant.) Nevertheless, this gets us close enough: we apply the action to $S$, obtaining both $rS$ (by definition) and $S - 1$ (because all the terms shift by one). Equating these yields

$$S = 1/(1 - r).$$

[3] (Engel exercise 11.5) Find all polynomials $p$ satisfying $p(x + 1) = p(x) + 2x + 1$.

**Solution.** The “$2x + 1$” on the right is reminiscent of the expansion of $(x + 1)^2$. We will make it look so by adding the missing $x^2$ to both sides:

$$p(x + 1) + x^2 = p(x) + (x + 1)^2.$$  

This is starting to look symmetric. An easy algebraic manipulation reveals an invariant:

$$p(x + 1) - (x + 1)^2 = p(x) - x^2.$$  

Now $q(x) = p(x) - x^2$ is also a polynomial and satisfies $q(x + 1) = q(x)$ for all $x$: it is invariant under shifting the argument $x$ by 1. Therefore (why?) $q$ must be a constant. Consequently $p(x) = x^2 + [\text{constant}]$.

**Comment.** The solution illustrates the *method of wishful thinking*. What would make a problem look easier? Maybe you can make it happen! Another example of this method is completing the square. (This simplifies $ax^2 + bx + c$ by trying to make it look like a perfect square. We cannot usually succeed, but in the process we come close enough: $ax^2 + bx + c = a(x + b/(2a))^2 - (b^2 - 4ac)/4a$ is a linear combination of a square and a constant. Using this method, you can readily compute $\int_{-\infty}^{\infty} \exp(-ax^2 - bx - c)dx$.)

[4] (Engel exercise 11.17, IMO 1968) Let $f$ be a real-valued function defined for all real numbers $x$ such that, for some positive constant $a$, the equation

$$f(x + a) = \frac{1}{2} + \sqrt{f(x) - f^2(x)}$$

holds for all $x$. Prove that the function $f$ is periodic; i.e., there exists a positive number $b$ such that $f(x + b) = f(x)$ for all $x$.

**Solution.** Clearing the square root (which is a natural simplifying move) gives

$$f^2(x + a) - f(x + a) + \frac{1}{4} = (f(x + a) - \frac{1}{2})^2 = [\sqrt{f(x) - f^2(x)}]^2 = f(x) - f^2(x).$$

It looks like $f^2(x) - f(x)$ could be manipulated into an invariant, so let’s simplify the notation by considering

$$g(x) = f^2(x) - f(x).$$

We have $g(x + a) = -g(x) - \frac{1}{4}$. Although the sign of $g$ has changed, we can apply this relation again to $g(x + a)$ to get a positive sign back again:
\[ g(x + 2a) = g(x + a + a) = -g(x + a) - \frac{1}{4} = g(x) + \frac{1}{4} - \frac{1}{4} = g(x). \]

That is, \( g \) is periodic with period \( b = 2a \). To show this implies \( f \) is periodic with period \( 2b \), we need to relate \( f \) to \( g \):

\[
\begin{align*}
g(x) &= f^2(x) - f(x) = (f(x) - \frac{1}{2})^2 - \frac{1}{4} \text{ implies} \\
f(x) &= \sqrt{g(x) - \frac{1}{4}} + \frac{1}{2}.
\end{align*}
\]

There is no ambiguity: we must always take the positive square root so that \( \sqrt{f(x) - f^2(x)} \) is defined. Therefore \( f \) is also periodic with period \( b \).

\[ \text{[5]} \quad \text{(Putnam 1987 B1)} \quad \text{Evaluate} \quad \int_{\frac{1}{2}}^{4} \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x) + \ln(x+3)}} \, dx. \]

**Solution.** The integrand is indeed Riemann integrable because it is well-defined and continuous. The presence of the 3’s and 9’s in the integrand and that the midpoint of the region of integration \([2, 4]\) is also 3 both suggest looking for some symmetry to simplify the integrand. Note that as \( x \) ranges from 2 to 4, \( x + 3 \) ranges from 5 to 7 and \( 9 - x \) ranges downwards from 7 to 5. The first operation to try, then, is replacing \( x \) by \( 6 - x \), which promisingly exchanges \( 9 - x \) and \( x + 3 \) and simply reverses the direction of integration without changing the region of integration. Upon making this substitution the integral becomes

\[ I = \int_{\frac{1}{2}}^{4} \frac{\sqrt{\ln(x+3)}}{\sqrt{\ln(x+3) + \ln(9-x)}} \, dx. \]

The denominator remains the same, crying out for us to add the old expression to the new:

\[
2I = I + I = \int_{\frac{1}{2}}^{4} \frac{\sqrt{\ln(x+3)} + \sqrt{\ln(9-x)}}{\sqrt{\ln(x+3) + \ln(9-x)}} \, dx = \int_{\frac{1}{2}}^{4} \, dx = 2,
\]

whence \( I = 1 \).

\[ \text{[6]} \quad \text{What is} \quad \sqrt[3]{-1 + \sqrt[3]{3 + \sqrt[3]{-1 + \sqrt[3]{3 + \cdots}}}}? \]

**Solution.** Perhaps the simplest approach is to consider \( L = \sqrt[3]{3 + \sqrt[3]{-1 + \sqrt[3]{3 + \cdots}}} \) instead and then compute \( \sqrt[3]{-1 + L} \) afterwards. Look at the operations that are going on: subtract one, take a cube root, add three, take a square root, and repeat ad infinitum. The composition of these four operations is \( x \to f(x) = \sqrt[3]{3 + \sqrt[3]{-1 + x}} \). If repetitions are to lead to a limit \( L \), then \( L \) must be an invariant of \( f \); \( f(L) = L \). Applying the inverse of \( f \) to both sides—square, subtract three, cube, add one—gives \((L^2 - 3)^3 + 1 = L\).
To solve this degree-six equation, it pays to reflect for a minute rather than diving in with a brute-force expansion. The graph of \((L^2 - 3)^3 - L + 1\) clearly has two real roots, one greater than \(\sqrt{3}\) and the other between \(-\sqrt{3}\) and 0. It is *always* worthwhile checking for a rational solution. This involves examining factors of the constant term, which is \((-3)^2 + 1 = -26\), divided by factors of the leading term, which is 1. We therefore plan to plug in \(\pm 1\), \(\pm 2\), and \(\pm 13\), in that order (from easiest to hardest to compute with). We stop with \(L = 2\) as a solution (it’s clear \(\pm 13\) is too large in size).

(Could there be other solutions? The negative real root must be irrational. Plugging in \(L = -1\) shows it is less than -1. This would imply \(\sqrt{3} + \sqrt[3]{-1 + \sqrt{3} + \cdots}\) is negative, which is contrary to the convention of taking the positive root. Also, an expression like this is real, not complex. Thus none of the other five roots is a valid candidate for the solution.)

Finally, we have to subtract one from \(L\) and take the cube root: the value is \(\sqrt[3]{-1 + 2} = 1\).

[7] All distances and angles are invariant under rotation and reflection in space. (This is the basis of about half of high school geometry. For instance, all the congruence theorems about triangles essentially assert that three distances or angles are invariants of a triangle’s orbit under the *Euclidean group* of rotations and reflections and that they determine the orbit uniquely.)

[8] (Zeitz 3.4.2) An invariant of the triple \(\{C, P, L\}\) where \(C\) is a plane circle, \(P\) is a point, and \(L\) is a line through the circle and the point is the *power of the point with respect to the circle*.

For instance, suppose we know \(P\) is distance 5 from the center \(O\) of \(C\), which has radius 3. A ray \(PX\) enters the interior of \(C\) at \(X\) at distance of 3 from \(P\) and leaves \(C\) again at \(Y\). What is the distance from \(P\) to \(Y\)?
Solution. The Pythagorean Theorem tells us the distance from P to C along a line through P and tangent to C is 4. Therefore the power of P with respect to C is $4 \times 4 = 16$. But also $16 = PX \cdot PY = 3 \cdot PY$, so $PY = 16/3$.

The Second law of thermodynamics states that the entropy of an isolated system will tend not to decrease over time: it is a mono-invariant. More quantitatively, an isolated thermodynamic system satisfies $dE - TdS + PdV \leq 0$ (variables are energy, temperature, entropy, pressure, volume). The integral therefore is a mono-invariant. Chemists use this fact to reason about the direction of chemical reactions, for instance.

The energy of a closed mechanical system is an invariant, or conserved quantity: it does not change over time as the system evolves. Using this fact often simplifies finding the solution to physical problems. For instance, consider Kepler’s problem of finding the motion of a particle in a central gravitational field.

Solution (Landau & Lifschitz, Mechanics, 3rd Ed., sections 14-15). Newton’s equations of motion are complicated: in spherical coordinates $(\rho, \theta, \phi)$, the gravitational potential $U$ depends on $\rho$ only, so the force $F$ has magnitude $|\partial U/\partial \rho|$ and is directed towards the center of the field, implying that the position $x(t) = (\rho(t), \theta(t), \phi(t))$ of a particle of mass $m$ satisfies

$$F = m\ddot{x}(t)/\dot{t}^2 = -\partial U/\partial \rho \ dr(t)/\rho$$

where $r$ is the vector from the origin to the position $x$. It is easier to solve these equations upon finding two invariants: the angular momentum $M$ (a vector with magnitude $M = mp^2d\theta/dt$) and the energy $E = \frac{1}{2}m[(\dot{\rho}/\dot{t})^2 + \rho^2(\dot{\phi}/\dot{t})^2] + U(\rho)$. The conservation of angular momentum implies all motion occurs in a plane perpendicular to $M$, so by directing the z-axis parallel to $M$, we may ignore the azimuthal coordinate $\phi(t)$. Rewriting the energy in terms of $M$ eliminates the other angular coordinate $\theta$, giving a relation between the radial coordinate $\rho$ and time $t$ in terms of a single integral:
\[ E = \frac{1}{2}m([d\rho/dt]^2 + M^2/(2\rho^3)) + U(\rho); \]

\[ d\rho/dt = \sqrt{\left(\frac{2}{m} \left[ E - U(\rho) - \frac{M^2}{2\rho^2} \right] \right)}; \]

\[ t = \frac{m}{\sqrt{2}} \int d\rho / \sqrt{E - U(\rho) - \frac{M^2}{2\rho^2}}. \]

The conservation of \( M \) gives a comparable integral formula for \( \theta \) in terms of \( \rho \):

\[ \theta = \sqrt{\frac{m}{2}} \int \frac{M\rho}{\rho^2 \sqrt{E - U(\rho) - \frac{M^2}{2\rho^2}}}. \]

For gravity, \( U(\rho) \) is proportional to \(-1/\rho\), say \( U(\rho) = -\alpha/\rho \) for a constant \( \alpha \). The integrals can be computed explicitly; in particular, by choosing the origin of \( \theta \) appropriately (the point of perihelion), the constant of integration in \( \theta \) disappears and we obtain

\[ \theta = \sqrt{\frac{m}{2}} \int \frac{M\rho}{\rho^2 \sqrt{E - U(\rho) - \frac{M^2}{2\rho^2}}} = \sqrt{\frac{m}{2}} \int \frac{M\rho}{\rho^2 \sqrt{E + \frac{\alpha}{\rho} - \frac{M^2}{2\rho^2}}}, \]

\[ \frac{M^2}{\rho} / m\alpha \cos(\theta). \]

This is the equation of an ellipse of eccentricity \( 1 + \sqrt{1 + 2EM^2 / m\alpha^2} \) having one focus at the origin.

**Exercises**

**Engel 1.11.** In the figure, you may invert the colors of all squares in any row, column, or a parallel to one of the diagonals. In particular, you may change the color of any individual corner square. Can you make all the squares the same color?

**Engel 1.25.** Let points \( A_1, A_2, \ldots, A_{2m} \) be vertices of a convex \( 2m \)-gon. \( P \) is a point in its interior that does not lie on any diagonal. Show that \( P \) lies inside an even number of the triangles determined by vertices among \( A_1, A_2, \ldots, A_{2m} \).
Engel 1.34. Is it possible to transform \( f(x) = x^2 + 4x + 3 \) into \( g(x) = x^2 + 10x + 9 \) by a sequence of transformations of the form 
\[
\begin{align*}
f(x) &\rightarrow x^2 f(1/x + 1) \quad \text{or} \quad f(x) \rightarrow (x - 1)^2 f(1/[x - 1])?
\end{align*}
\]

Zeitz example 3.4.9. Let \( P_1, P_2, \ldots, P_{1997} \) be distinct points in the plane. Connect the points with the line segments \( P_1P_2, P_2P_3, P_3P_4, \ldots, P_{1996}P_{1997}, \) and \( P_{1997}P_1. \) Can one draw a line that passes through the interior of every one of these segments?

Zeitz 3.4.34. The first six terms of a sequence are 0, 1, 2, 3, 4, 5. Each subsequent term is the last digit of the sum of the six previous terms. In other words, the seventh term is 5 because \( 0+1+2+3+4+5 = 15, \) the eighth term is 0 because \( 1+2+3+4+5+5 = 20, \) etc. Can the subsequence \( \ldots, 1, 3, 5, 7, 9, \ldots \) occur anywhere?

Zeitz 2.4.24. Several marbles are placed on a circular track. The radii of the marbles are negligibly small. Each marble is randomly given an orientation, clockwise or counterclockwise. At time zero, each marble begins to travel with a constant speed that would take it entirely around the track in one minute, where the direction of travel depends on the orientation. However, whenever two marbles collide, they bounce back with no change in speed, obeying the laws of inelastic collision. What can you say about the possible locations of the marbles after one minute, with respect to their original positions? There are three factors to consider: the number of marbles, their initial locations, and their initial orientations.

Engel 11.4. Show that a function \( f \) is periodic when, for fixed \( a \) and any \( x, \)
\[
f(x + a) = (1 + f(x)) / (1 - f(x)).
\]

Engel 11.6. Find all functions \( f \) which are defined for all \( x \in \mathbb{R} \) and, for any \( x, y, \) satisfy
\[
xf(y) + yf(x) = (x + y)f(x)f(y).
\]

Landau & Lifschitz (14.11). When the magnitude of angular momentum \( M \) is nonzero, a particle cannot fall all the way to the center of a gravitational field (because the integral expressing time \( t \) in terms of distance \( r \) diverges as \( r \) goes to zero). For what forms of potential energy \( U(r) \) is it possible for particles to fall to the center, even with nonzero angular momentum?

Warm-up 1. Let \( p(x) = p_0 + p_1 x + \ldots + p_d x^d \) and \( q(x) \) be polynomials with complex coefficients. Assume the degree of \( q \) does not exceed \( d - 2. \) Let \( \{x_1, x_2, \ldots, x_d\} \) be the solutions to \( p(x) = q(x), \) counted as usual with their multiplicity. Show that \( x_1 + x_2 + \ldots + x_d \) depends on \( p(x) \) but not on \( q(x). \)

Warm-up 2. Let \( P \) be a linear transformation of \( \mathbb{R}^d \) to itself. Prove (a) the sum of diagonal elements of \( P \) with respect to any basis of \( \mathbb{R}^d \) is invariant (and therefore is a property of \( P \) itself, called its trace and written \( tr(P) \)) and that (b) when \( P^2 = P, \) \( tr(P) = \operatorname{rank}(P) = \) the dimension of the image of \( P. \)
Problems

Do not let the difficulties of these problems (as determined by the actual contest results) fool you: some of those found to be hard or very hard by the contestants have simple, straightforward solutions. I have listed them in a subjective order of ascending difficulty.

**Putnam 1977 A1.** Consider all lines that meet the graph of
\[ y = 2x^4 + 7x^3 + 3x - 5 \]
in four distinct points, say \((x_i, y_i), i = 1, 2, 3, 4\). Show that
\[ (x_1 + x_2 + x_3 + x_4)/4 \]
is independent of the line and find its value.

**Putnam 1986 B6.** Suppose \(A, B, C, D\) are \(n \times n\) matrices with entries in a field \(F\), satisfying the conditions that \(AB^t\) and \(CD^t\) are symmetric and \(AD^t - BC^t = I\). Here \(I\) is the \(n \times n\) identity matrix, and if \(M\) is an \(n \times n\) matrix, \(M^t\) is the transpose of \(M\). Prove that \(A^t D - C^t B = I\). [Very hard]

**Putnam 1985 B5.** Evaluate \(\int_{0}^{\infty} t^{4} e^{-1985(t+1)} dt\). You may assume that \(\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\). [Moderately hard]

**Putnam 1993 A5.** Show that
\[
\int_{-10}^{10} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{1/10}^{11/10} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{10/101}^{11/101} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx
\]
is a rational number. [Hard]

**Putnam 1985 B6.** Let \(G\) be a finite set of real \(n \times n\) matrices \(\{M_i\}, 1 \leq i \leq r\), that form a group under matrix multiplication. Suppose that \(\sum_{i=1}^{r} tr(M_i) = 0\), where \(tr(A)\) denotes the trace of the matrix \(A\). Prove that \(\sum_{i=1}^{r} M_i\) is the \(n \times n\) zero matrix. [Hard]

**Putnam 1988 A6.** If a linear transformation \(A\) on an \(n\)-dimensional vector space has \(n + 1\) eigenvectors such that any \(n\) of them are linearly independent, does it follow that \(A\) is a scalar multiple of the identity? Prove your answer. [Moderately easy].

**Putnam 1995 B4.** Evaluate \(\sqrt{2207 - \frac{1}{2207 - \frac{1}{2207 - \ldots}}} \). Express your answer in the form \(\frac{a + b\sqrt{c}}{d}\), where \(a, b, c, d\) are integers. [Moderate]

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